$S = (S^1_t, S^2_t, \ldots, S^n_t, \mathbf{B}_t)_{t \geq 0}$

are the tradable assets

$\lambda = (\lambda^1_t, \lambda^2_t, \ldots, \lambda^n_t, \beta_t)_{t \geq 0}$ is a trading strategy

$V_{t_n} = \lambda_{t_{n-1}} \cdot S_{t_n}$

$V_{t_n} = \lambda_{t_n} \cdot S_{t_n}$

$\implies (\lambda_{t_n} - \lambda_{t_{n-1}}) \cdot S_{t_n} = 0$

$\implies S_{t_{n-1}} + (S_{t_n} - S_{t_{n-1}})$

$\Delta_n \lambda \implies \Delta_n S$

$d \mathbb{P} [\lambda, S]_t$

$\Delta_n \lambda$

$\mathbb{E} \left[ \sum_{n=1}^{\infty} g_{t_{n-1}} (W_{t_n} - W_{t_{n-1}}) \right]$

$dX_t = g_t \, dW_t$

$\mathbb{E} \left[ \sum_{n=1}^{\infty} (W_{t_n} - W_{t_{n-1}})^2 \right] = t$ a.s.
\[ [X, Y]_t = \lim_{\Pi_{tn} \to 0} \sum_{n=1}^{\Pi_{tn}} \frac{1}{2} (X_{tn} - X_{tn-1})(Y_{tn} - Y_{tn-1}) \]

Self-financing trading strategies satisfy

\[ d[V, S]_t + S_t \cdot d\lambda_t = 0 \]

Value process

\[ V_t = (V_t)_{t \geq 0}, \quad V_t = \lambda_t \cdot S_t \]

\[ dV_t = \lambda_t \cdot S_t + \lambda_t \cdot dS_t + d[V, S]_t \]

\[ \text{Itô's product rule} \]

for self-financing strategies

\[ dV_t = \lambda_t \cdot dS_t \]

An arbitrage trading strategy is one which is self-financing s.t.

i) \[ V_0 = 0 \]

ii) \[ \exists t \text{ s.t.} \quad \begin{align*}
& a) P(V_t \geq 0) = 1 \\
& b) P(V_t > 0) > 0
\end{align*} \]
Black-Scholes Model

i) $S = (S_t)_{t \geq 0}$, a tradable risky asset satisfying the SDE

$$dS_t = S_t \mu dt + S_t \sigma dW_t, \quad \sigma > 0, \mu \in \mathbb{R}$$

Geometric Brownian Motion (GBM)

$$S_t = S_0 + \int_0^t S_u \mu du + \int_0^t S_u \sigma dW_u$$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t = d \left( \log S_t + \sigma W_t \right)$$

$W = (W_t)_{t \geq 0}$ is a 1D Brownian motion

ii) Risk-free bank account $B = (B_t)_{t \geq 0}$

$$B_t = e^{rt}, \quad \text{i.e. interest rates are constant}$$

$$dB_t = r B_t dt, \quad B_0 = 1$$

iii) $F = (F_t)_{t \geq 0}$ is a tradable asset that is a European styled contingent claim:

$$F_T = F(S_T), \quad \text{i.e.} \quad F(S) = (S - K)^+$$

$F_{mt}$ is an $F$-martingale

we will assume that $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t.

$$F_t = F(t, S_t)$$
i.e. \( F \) in Markovian in the underlying asset price.

\[
f(t, s) = s^2 - 4t^2 s
\]
\[ dS_t = S_t \mu \, dt + S_t \sigma \, dW_t \]
\[ dB_t = r \, B_t \, dt \]
\[ f_t = f(t, S_t) \quad \text{we do know that } f(T_s) = f(s) \]

Further assume \( f \in C^{1,2} \) i.e. differentiable in \( t \) twice and in \( s \).

Take a self-financing portfolio \( \gamma \)
\[ \pi_t = (\alpha_t, \beta_t, -1) \quad \text{set } V_0 = 0 \]
\[ V_t = \alpha_t \, S_t + \beta_t \, B_t + \gamma_t \, f_t \]
\[ dV_t = \alpha_t \, dS_t + \beta_t \, dB_t + \gamma_t \, df_t \]

Self-financing condition
\[ = \alpha_t \, dS_t + \beta_t \, dB_t - df_t \]
\[ = \alpha_t \left( S_t \mu \, dt + S_t \sigma \, dW_t \right) + r \beta_t \, B_t \, dt \]
\[ - df_t \]

From Itô's lemma we therefore have:
\[ df_t = df(t, S_t) \]
\[ = \left[ \partial_{t} f(t, S_t) + \mu S_t \partial_{S} f(t, S_t) + \frac{1}{2} \sigma^2 S_t^2 \partial_{SS} f(t, S_t) \right] \, dt \]
\[ + \sigma S_t \partial_{S} f(t, S_t) \, dW_t \]
\[ \Delta f_t = f_{t+\Delta t} - f_t \]
\[ = f(t+\Delta t, s_{t+\Delta t}) - f(t, s_t) \]
\[ = f(t+\Delta t, s_t + \Delta s_t) - f(t, s_t) \]
\[ = \frac{\partial f(t, s_t)}{\partial t} \Delta t + \frac{\partial f(t, s_t)}{\partial s} \Delta s_t + \frac{1}{2} \sigma^2 \frac{\partial^2 f(t, s_t)}{\partial s^2} (\Delta s_t)^2 + \ldots \]
\[ = \sigma^2 (\Delta s_t^2 + \cdots) \]

\[ \Rightarrow \Delta V_t = \mathbb{E}_t \left( \alpha_t s_t \Delta t + \beta_t B_t - (\partial_t + L_t) f(t, s_t) \right) \Delta t \]
\[ + \left( \alpha_t s_t \sigma - \sigma s_t \partial_s f(t, s_t) \right) \Delta W_t \]

Locally remove risk by setting \( \alpha_t = 0 \).

\[ \Rightarrow \alpha_t = \partial_s f(t, s_t) \]

\[ \Rightarrow \Delta V_t = A_t \Delta t \]

To avoid arbitrage we must have \( A_t = 0 \).

\[ \Rightarrow \Delta V_t = 0 \Rightarrow V_t = 0 . \]

\[ \alpha_t s_t + \beta_t B_t - f_t \]

\[ \Rightarrow \beta_t B_t = f_t - \alpha_t s_t \]

\[ \Rightarrow \partial_s f(t, s_t) \left( \alpha_t s_t \Delta t + r (f_t - \alpha_t s_t) - (\partial_t + L_t) f(t, s_t) \right) = 0 \]
\[ (\partial_t + L_t)f(t, s_t) = (\mu - r) s_t \partial_s f(t, s_t) + r f(t, s_t) \]

\[ m s_t \partial_s f + \frac{1}{2} \sigma^2 s_t^2 \partial_s^2 f = 0 \]

must hold true for paths

\[ (\partial_t + r s_t \partial_s + \frac{1}{2} \sigma^2 s_t^2 \partial_s^2) f(t, s) = r f(t, s), \]

\[ f(T, s) = F(s) \]

is the Black-Scholes Partial Differential Equation (PDE)
A simple version of BS PDE:

\[
\begin{align*}
(\partial_t + \frac{1}{2} \sigma^2 \partial_{xx}) h(t, x) &= 0 \\
h(T, x) &= H(x)
\end{align*}
\]

where \( \sigma \) is the volatility.

**Numerical Solution:**

- \( h_{n+1, j} \) is a simple discretization of the PDE.

\[
\frac{\partial}{\partial t} h(t_n, x_j) = \frac{h_{n+1,j} - 2h_{n,j} + h_{n-1,j}}{(\Delta x)^2} + \frac{1}{2} \sigma^2 \frac{h_{n+1,j} - 2h_{n,j} + h_{n-1,j}}{(\Delta x)^2}
\]

\[
\frac{\partial}{\partial x} h(t_n, x_j) = \frac{h_{n,j+1} - h_{n,j-1}}{2\Delta x} + \frac{1}{2} \sigma^2 \frac{h_{n+1,j} - 2h_{n,j} + h_{n-1,j}}{(\Delta x)^2}
\]

\[
h_{n+1, j} = h_{n-1, j} + \frac{1}{2} \sigma^2 \frac{h_{n+1, j} - 2h_{n, j} + h_{n-1, j}}{(\Delta x)^2} + \frac{1}{2} \sigma^2 \frac{h_{n+1, j} - 2h_{n, j} + h_{n-1, j}}{(\Delta x)^2}
\]

\[
h_{n, j} = a_1 h_{n+1, j} + a_2 h_{n, j} + a_3 h_{n-1, j}
\]

where \( a_1, a_2, a_3 \) are coefficients determined by the specific discretization scheme.